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ON CHAINS OF PRIME SUBMODULES

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ABSTRACT

In this paper, we study the dimension of a module over a commutative ring, which is defined to be the length of a longest chain of prime submodules. This notion is analogous to the usual Krull dimension of a ring. We investigate how some bounds on the dimension of modules are related to the structure of the underlying ring. The dimension of finitely generated modules over a Dedekind domain is also studied. By examining the structure of prime submodules, a formula for the dimension of a free module of finite rank, over a Noetherian one-dimensional domain, is obtained.

1. Introduction

Let R be a commutative ring with identity and let M be a unital R-module. A submodule P of M is called a **prime** submodule of M if

(i) $P \neq M$, and

(ii) whenever $r \in R$ and $m \in M \backslash P$ with $rm \in P$, then $rM \subseteq P$.

Note that if P is a prime submodule of M, then $(P : M)$, the annihilator of M/P over R, is a prime ideal. In that case, we say P is a $(P : M)$-prime

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submodule of M. Prime submodules have been studied extensively by many authors (for example, see [1], [2], [5], [7], [8], [10], [11], and [12]).

Suppose that the module M contains a prime submodule P. Then the **height** of P, denoted by $\operatorname{ht} P$, is the greatest non-negative integer n such that there exists a chain of prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P,$$

and $\operatorname{ht} P = \infty$ if no such n exists. The dimension $D(M)$ of M is defined by

$$D(M) = \sup\{\operatorname{ht} P : P \text{ is a prime submodule of } M\}.$$

$D(R)$ is just the usual (Krull) dimension of R. In this paper, we investigate how the bounds on the dimension of modules are related to the structure of the underlying ring. By characterizing prime submodules of $R^{(n)}$, we are able to work out an explicit formula for $D(R^{(n)})$, where R is a Noetherian one-dimensional domain. There are already some known results (see [1] and [10]) about the dimension of modules. In [1], chains of certain prime modules were studied. A lower bound for $D(M)$ was given in [10].

Before we describe the main results of this paper, we first fix some notation. All the rings in this paper are assumed to be commutative with identity, not necessarily Noetherian, and modules will be unital modules. A local ring means a commutative Noetherian ring with a unique maximal ideal.

Let R be a ring and M be an R-module. It is clear that

$$D(M) = \sup\{D(M/\mathfrak{P}M) : \mathfrak{P} \text{ is a prime ideal of } R\}.$$

In view of this, we shall work mostly with domains.

Let R be a domain. We shall use $R^{(n)}$ to denote the free R-module of rank n, where n is a positive integer. Sometimes we write $R \oplus R$ instead of $R^{(2)}$. Let G and H be submodules of an R-module M. We define $(G : H)$ to be the ideal

$$\{r \in R : rH \subseteq G\}.$$

If N is an R-submodule of M, then $(N : M)$ is the annihilator of the quotient R-module M/N. In particular, the annihilator of M is $(0 : M)$. Sometimes, we write $\operatorname{ann}(M)$ instead of $(0 : M)$. If M is torsion-free, then the rank of M is the dimension of the vector space $FM = F \otimes_R M$ over F, where F is the field of fractions of R. The rank of M is denoted by $\operatorname{rk} M$. If the base domain is not clear, we shall use $\operatorname{rk}_R M$ to denote the rank of M over R.

Let R be a ring, not necessarily a domain, and $\mathfrak{P}$ be a prime ideal of R. Let M be an R-module. We define $K_{\mathfrak{P}}(M)$ to be the following R-submodule of M:

$$\{m \in M : cm \in \mathfrak{P}M \text{ for some } c \in R \setminus \mathfrak{P}\}.$$

It is easily shown that $M = K_{\mathfrak{P}}(M)$ or $K_{\mathfrak{P}}(M)$ is a $\mathfrak{P}$-prime submodule of M. Hence $M = K_{\mathfrak{P}}(M)$ or $M/K_{\mathfrak{P}}(M)$ is a non-zero torsion-free $R/\mathfrak{P}$-module. The $\mathfrak{P}$-rank of M, denoted by $\operatorname{rk}_{\mathfrak{P}} M$, is defined by

$$\operatorname{rk}_{\mathfrak{P}} M = \operatorname{rk}_{R/\mathfrak{P}}(M/K_{\mathfrak{P}}(M)).$$

In case R is a domain and M is an R-module, $K_0(M)$ is the torsion submodule of M, and $\operatorname{rk}_0 M = \operatorname{rk}_R(M/K_0(M))$. We use $\mu(M)$ to denote the least number of generators required to generate M. Let $\mathfrak{P}$ be a maximal ideal of R. We shall denote by $\nu(R_{\mathfrak{P}})$ the least positive integer k such that every ideal of $R_{\mathfrak{P}}$ can be generated by k elements. If no such k exists, we put $\nu(R_{\mathfrak{P}}) = \infty$. Next we define

$$\nu(R) = \sup\{\nu(R_{\mathfrak{P}}) : \mathfrak{P} \text{ is a maximal ideal of } R\}.$$

Now we have all the necessary notation to describe the main results of this paper.

As mentioned earlier, there is a lower bound for $D(M)$ (see [10]). It is natural to ask whether there is any upper bound for $D(M)$. It turns out that there is and one is given in Corollary 2.8, namely $D(M) \leq \mu(M)D(R) + \mu(M) - 1$. In general, this bound can be strict (see Theorem 6.1). In order to see whether the above upper bound can be improved, we restrict R to be a Prüfer domain. For any finitely generated module M over a Prüfer domain R, we have the following sharper upper bound for $D(M)$ (see Corollary 3.5):

$$D(M) \leq D(R) + \mu(M) - 1.$$

The converse holds when R is Noetherian. In fact, for a Noetherian domain R, the following statements are equivalent (see Theorem 3.6):

(i) R is a Dedekind domain.

(ii) $D(R^{(n)}) = D(R) + n - 1$ for any positive integer n.

(iii) $D(M) \leq D(R) + \mu(M) - 1$ for any finitely generated R-module M.

(iv) $D(R \oplus R) \leq D(R) + 1$.

In section 4, we carry out a detailed study of dimension of finitely generated modules over a Dedekind domain. There is another characterization of Dedekind domains in terms of the dimension of the finitely generated modules. It turns out that for a Noetherian domain R to be Dedekind, it is necessary and sufficient that the dimension of every finitely generated R-module M must be given by $D(M) = \sup\{D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}} M - 1 : \mathfrak{P}$ is a prime ideal of R and $\mathrm{ann}(M) \subseteq \mathfrak{P}\}$ (see Theorem 4.7).

Let R be a Noetherian one-dimensional domain. If M is a finitely generated torsion-free R-module, then one would expect $D(M)$ to depend on the structure of M and properties of R. However, if the module M is the free R-module $R^{(n)}$, for some positive integer n, then $D(M)$ ought to depend only on n and R. We shall show that this is indeed the case. More precisely, $D(R^{(n)})$ is given by (see Theorem 5.1):

$$D(R^{(n)}) = \begin{cases} 2n - \frac{n}{\nu(R)} & \text{if } \nu(R) \mid n, \\ 2n - [\frac{n}{\nu(R)}] - 1 & \text{if } \nu(R) \nmid n. \end{cases}$$

2. Bounds on the dimension

In the first half of this section, we look at some bounds for the dimension of modules. Later, we study the structure of 0-prime submodules of a free module over a domain. The following result ([10, Theorem 3.4]) gives a lower bound for $D(M)$.

PROPOSITION 2.1: *Let R be a domain and let M be a non-zero finitely generated torsion-free R-module. Then $D(M) \geq D(R) + \mathrm{rk}\, M - 1$.*

Note that if R is any domain with field of fractions $K \neq R$ and M is the R-module K, then 0 is the only prime submodule of M so that $D(M) = \mathrm{ht}\, 0 = 0$, $\mathrm{rk}\, M = 1$ and $D(R) \geq 1$, so that $D(M) < D(R) + \mathrm{rk}\, M - 1$. Thus, Proposition 2.1 requires the module M to be finitely generated. Next we generalize Proposition 2.1.

PROPOSITION 2.2: *Let R be a ring and let M be a non-zero finitely generated R-module. Then $D(M) \geq \sup\{D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}} M - 1 : \mathfrak{P}$ is a prime ideal of R and $\mathrm{ann}(M) \subseteq \mathfrak{P}\}$.*

Proof: Let $\mathfrak{P}$ be any prime ideal of R such that $\mathrm{ann}(M) \subseteq \mathfrak{P}$ with $K_{\mathfrak{P}}(M)$ as defined in section 1. Note that $M/K_{\mathfrak{P}}(M)$ is a torsion-free $(R/\mathfrak{P})$-module. Suppose that $M = K_{\mathfrak{P}}(M)$. Since M is finitely generated, it follows that $dM = 0$ for some $d \in R \setminus \mathfrak{P}$ (see [1, Corollary 1.2]), and hence $d \in \mathrm{ann}(M) \subseteq \mathfrak{P}$, a contradiction. Hence $M \neq K_{\mathfrak{P}}(M)$. Clearly,

$$\begin{aligned} D(M) \geq & D(M/K_{\mathfrak{P}}(M)), \\ \geq & D(R/\mathfrak{P}) + \mathrm{rk}_{R/\mathfrak{P}}(M/K_{\mathfrak{P}}(M)) - 1, \text{ by Proposition 2.1,} \\ \geq & D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}} M - 1. \end{aligned}$$

The result follows. ∎

We shall see in Corollary 4.6 that the lower bound in Proposition 2.2 is attained for finitely generated modules over a Dedekind domain. It will be proved shortly that Proposition 2.2 holds when the supremum is taken over all maximal ideals, not necessarily containing $\mathrm{ann}(M)$. To do that, parts of the following result will be needed.

PROPOSITION 2.3: *Let R be a ring, $\mathfrak{M}$ be a maximal ideal of R, and M be a non-zero finitely generated R-module. Then the following statements hold.*

(i) $K_{\mathfrak{M}}(M) = \mathfrak{M}M$.

(ii) $\mathrm{ann}(M) \subseteq \mathfrak{M}$ *if and only if* $\mathfrak{M}M \neq M$.

(iii) $\mathrm{rk}_{\mathfrak{M}} M = \mu(M/\mathfrak{M}M)$.

If, in addition, R is a domain but not a field, then $D(M) \geq \mathrm{rk}(M/T) = \mathrm{rk}_0 M$, where $T = K_0(M)$ is the torsion submodule of M.

Proof: The necessary part of (ii) follows from an argument in the proof of [1, Corollary 1.2] or Nakayama's Lemma. The other assertions are easy consequences of the definitions. ∎

COROLLARY 2.4: *Let R be a ring and let M be a non-zero finitely generated R-module. Then*

$$D(M) \geq \sup\{\mu(M/\mathfrak{M}M) - 1 : \mathfrak{M} \text{ is a maximal ideal of } R\}.$$

Proof: Let $\mathfrak{M}$ be a maximal ideal of R. If $M = \mathfrak{M}M$, then $\mu(M/\mathfrak{M}M) = \mu(0) = 0$. So that $D(M) \geq \mu(M/\mathfrak{M}M) - 1$. Now, suppose that $M \neq \mathfrak{M}M$. By Proposition 2.3 (ii), $\mathrm{ann}(M) \subseteq \mathfrak{M}$. It follows from Proposition 2.2 that $D(M) \geq D(R/\mathfrak{M}) + \mathrm{rk}_{\mathfrak{M}} M - 1$. As $\mathfrak{M}$ is maximal, $D(R/\mathfrak{M}) = 0$. Note that, by Proposition 2.3 (iii), $\mathrm{rk}_{\mathfrak{M}} M = \mu(M/\mathfrak{M}M)$. Consequently, $D(M) \geq \mu(M/\mathfrak{M}M) - 1$. ∎

Definition 2.5: Let $\mathfrak{P}$ be a prime ideal of a ring R and M be an R-module. A chain of prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_k$$

is homogeneous if each of $P_0, P_1, \ldots, P_k$ is a $\mathfrak{P}$-prime submodule of M.

Suppose that P and Q are $\mathfrak{P}$ and $\mathfrak{Q}$-prime submodules of M, respectively, with $P \subseteq Q$. Then $\mathfrak{P} \subseteq \mathfrak{Q}$. From this observation, we see that a chain of prime submodules of M is made up of homogeneous chains of prime submodules.

LEMMA 2.6: *Let M be a finitely generated torsion-free module over a domain R. Then the length of any homogeneous chain of prime submodules of M is at most $\mu(M) - 1$.*

Proof: Let

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_k$$

be a homogeneous chain of $\mathfrak{P}$-prime submodules of M. Note that $M/P_0, P_1/P_0, P_2/P_0, \ldots, P_k/P_0$ are torsion-free modules over $R/\mathfrak{P}$. As M is finitely generated, M/P_0 is also finitely generated. Hence $P_1/P_0, P_2/P_0, \ldots, P_k/P_0$ all have finite rank over $R/\mathfrak{P}$. Now

$$\begin{aligned} \mathrm{rk}_{R/\mathfrak{P}}\, M/P_0 &= \mathrm{rk}_{R/\mathfrak{P}}\, M/P_k + \sum_{i=1}^{k} \mathrm{rk}_{R/\mathfrak{P}}\, P_i/P_{i-1} \\ &\geq k + 1. \end{aligned}$$

Clearly, $\mathrm{rk}_{R/\mathfrak{P}}\, M/P_0 \leq \mu(M/P_0) \leq \mu(M)$. Therefore $k \leq \mu(M) - 1$. ∎

THEOREM 2.7: *Let M be a finitely generated torsion-free module over a domain R. Then $D(M) \leq \mu(M)D(R) + \mu(M) - 1$. In particular, $D(R^{(n)}) \leq nD(R) + n - 1$.*

Proof: The last assertion follows immediately from the first. We now prove the first part.

If $D(R)$ is infinite, then there is nothing to prove. We may assume that $D(R) \geq 0$ is finite. By the remark after Definition 2.5, any chain of prime submodules of M is made up of at most $D(R) + 1$ distinct homogeneous chains of prime submodules — each homogeneous chain corresponding to a different prime ideal. By Lemma 2.6, the length of each homogeneous chain is at most $\mu(M) - 1$. Hence each homogeneous chain has at most $\mu(M)$ terms. Then the number of terms in a chain of prime submodules of M is at most $(D(R) + 1)\mu(M)$. Therefore the length of a chain of prime submodules is at most $(D(R) + 1)\mu(M) - 1$. ∎

COROLLARY 2.8: *Let M be a finitely generated module over a domain R. Then $D(M) \leq \mu(M)D(R) + \mu(M) - 1$.*

Proof: Suppose that $\mu(M) = k$. Then M is a quotient module of the free module $R^{(k)}$. Thus $D(M) \leq D(R^{(k)})$. The required result follows from Theorem 2.7. ∎

We shall give an example in section 6 (see Theorem 6.1) to show that in certain cases in Corollary 2.8, $D(M) = \mu(M)D(R) + \mu(M) - 1$.

In order to determine $D(R^{(n)})$, we need to know something about the structure of prime submodules of $R^{(n)}$. In the rest of this section, we shall investigate the structure of prime submodules of $R^{(n)}$.

LEMMA 2.9: *Let R be a domain, let n and k be positive integers, let M denote the free R-module $R^{(n)}$, and let N be a submodule of M of rank $n-k$. Then N is a 0-prime submodule of M if and only if there exist a non-zero $k \times n$ echelon matrix (a_{ij}) over R and a basis $\{u_1, u_2, \ldots, u_n\}$ of M such that*

$$N = \{r_1u_1 + r_2u_2 + \cdots + r_nu_n \in M : \sum_{j=1}^{n} a_{ij}r_j = 0 \text{ for } i = 1, 2, \ldots, k\}.$$

Moreover, in this case $\operatorname{ht} N = n - k$.

Proof: The sufficiency part is clear. We now prove the necessary part.

Suppose that N is a 0-prime submodule of M. Note that M/N is torsion-free. We have a short exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\pi} M/N \longrightarrow 0,$$

where α and π are the natural injection and surjection, respectively. Then $\operatorname{rk} M/N = n - \operatorname{rk} N = k$. If $x \in M$, then we denote the image of x in M/N by $\bar{x}$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of M. Then $\{\bar{e_1}, \bar{e_2}, \ldots, \bar{e_n}\}$ is a generating set of M/N. After renumbering those $\bar{e_i}$'s, we may assume that

(a) $\{\bar{e_1}, \bar{e_2}, \ldots, \bar{e_k}\}$ is linearly independent over R, and

(b) for each $j = k+1, k+2, \ldots, n$, $\{\bar{e_1}, \bar{e_2}, \ldots, \bar{e_k}, \bar{e_j}\}$ is linearly dependent over R.

From (b), for each $j = k+1, k+2, \ldots, n$, there exists a non-zero $d_j \in R$ with $d_j\bar{e_j} \in \bigoplus_{i=1}^{k} R\bar{e_i}$. Put $d = d_{k+1}d_{k+2}\cdots d_n$. Note that $d \neq 0$ and $d(M/N) \subseteq \bigoplus_{i=1}^{k} R\bar{e_i}$. Define a map

$$\beta\colon M/N \longrightarrow \bigoplus_{i=1}^{k} R\bar{e_i}\colon \bar{x} \mapsto d\bar{x}.$$

It is clear that β is a well defined R-module homomorphism. As M/N is torsion-free, β is injective. Then we have an exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta\pi} \bigoplus_{i=1}^{k} R\bar{e_i}.$$

Let A be the $k \times n$ matrix representing $\beta\pi$ with respect to the bases $\{e_1, e_2, \ldots, e_n\}$ and $\{\bar{e_1}, \bar{e_2}, \ldots, \bar{e_k}\}$. It is clear that A is in echelon form. In fact $A = [dI_k|B]$, where I_k is the $k \times k$ identity matrix and B is an $k \times (n-k)$ matrix over R. Since $N = \ker \beta\pi$, it follows that N has the required form.

We now prove the moreover part. We have $\operatorname{rk} N = n - k$. In other words, FN has dimension $n-k$ over F, the field of fractions of R. Hence there exists a chain

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-k} = FN$$

of subspaces of FM. It follows that

$$0 = V_0 \bigcap M \subsetneq V_1 \bigcap M \subsetneq V_2 \bigcap M \subsetneq \cdots \subsetneq V_{n-k} \bigcap M = FN \bigcap M = N$$

is a chain of 0-prime submodules of M. Thus $\operatorname{ht} N \geq n - k$. On the other hand, if

$$0 = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_t = N$$

is a chain of prime submodules of M, then $(N_i : M) \subseteq (N : M) = 0$ so that M/N_i is torsion-free for all $0 \leq i \leq t$. Hence

$$0 = FN_0 \subsetneq FN_1 \subsetneq FN_2 \subsetneq \cdots \subsetneq FN_t = FN$$

is a chain of subspaces of FN, which gives $t \leq \operatorname{rk} N = n - k$. Therefore $\operatorname{ht} N = n - k$. ∎

Suppose that R is a domain. By Proposition 2.1, we have $D(R^{(n)}) \geq D(R) + n - 1$. If R has a prime ideal $\mathfrak{P}$ such that $\nu(R_{\mathfrak{P}})$ is sufficiently large, then the following proposition gives a sharper lower bound for $D(R^{(n)})$.

PROPOSITION 2.10: *Let R be a domain with elements $a_i \in R(1 \leq i \leq n)$ and a prime ideal $\mathfrak{P}$ such that*

$$\sum_{i=1}^{n} (I_i : Ra_i) \subseteq \mathfrak{P},$$

where $I_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} Ra_j$. Then $D(R^{(n)}) \geq n + D((R/\mathfrak{P})^{(n)}) \geq 2n + D(R/\mathfrak{P}) - 1$.

Proof: Let $M = R^{(n)}$. Put

$$P = \{(r_1, r_2, \ldots, r_n) \in M : a_1 r_1 + a_2 r_2 + \cdots + a_n r_n = 0\}.$$

Note that $a_i \neq 0 (1 \leq i \leq n)$. By Lemma 2.9, P is a 0-prime submodule of M with $\operatorname{ht} P = n-1$. Moreover, $P \subsetneq \mathfrak{P}M$. Thus

$$D(M) \geq \operatorname{ht} \mathfrak{P}M + D((R/\mathfrak{P})^{(n)}),$$
$$\geq n + D((R/\mathfrak{P})^{(n)}).$$

By Proposition 2.1, we have $D((R/\mathfrak{P})^{(n)}) \geq D(R/\mathfrak{P}) + n - 1$. Hence $n + D((R/\mathfrak{P})^n) \geq 2n + D(R/\mathfrak{P}) - 1$. ∎

Note that, in the statement of Proposition 2.10, the assumption on $\mathfrak{P}$ is equivalent to $\nu(R_{\mathfrak{P}}) \geq n$. We now end this section with a characterization of a prime submodule P in $R^{(n)}$ with $\operatorname{Ann}(R^{(n)}/P)$ being a maximal ideal.

Lemma 2.11: *Let R be a ring, let $\mathfrak{M}$ be a maximal ideal of R and let M be the free R-module $R^{(n)}$ for some positive integer n. Then K is an $\mathfrak{M}$-prime submodule of M if and only if there exist an integer $1 \leq k \leq n$ and a basis $\{m_1, m_2, \ldots, m_n\}$ of M such that $K = \mathfrak{M}m_1 + \cdots + \mathfrak{M}m_k + Rm_{k+1} + \cdots + Rm_n$.*

Proof: The sufficiency part is clear. We now prove the necessary part. Suppose that K is an $\mathfrak{M}$-prime submodule of M. If $n = 1$, then $M = Rm$ and $K = \mathfrak{M}m$ for some $m \in M$. Suppose that $n > 1$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of M. For each $1 \leq i \leq n$, let $\pi_i \colon M \longrightarrow R$ be the homomorphism defined by $\pi_i(r_1, r_2, \ldots, r_n) = r_i$, where $(r_1, r_2, \ldots, r_n) \in M$. If $\pi_i(K) \subseteq \mathfrak{M}$ for all $1 \leq i \leq n$, then $\mathfrak{M}M \subseteq K \subseteq \pi_1(K)e_1 + \cdots + \pi_n(K)e_n \subseteq \mathfrak{M}M$, so that $K = \mathfrak{M}M = \mathfrak{M}e_1 + \cdots + \mathfrak{M}e_n$.

Now suppose that $\pi_j(K) \not\subseteq \mathfrak{M}$ for some $1 \leq j \leq n$. Since $\mathfrak{M}M \subseteq K$, it follows that $\mathfrak{M} \subseteq \pi_j(K)$ and hence $\pi_j(K) = R$. There exists an element m_1 of K such that $\pi_j(m_1) = 1$ and hence $\{m_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n\}$ is a basis of M. Moreover, if $L = \sum_{i=1, i\neq j}^{n} Re_i$, then L is a free R-module of rank $n-1$ and $K = Rm_1 \oplus (K \cap L)$. Note that $K \cap L$ is an $\mathfrak{M}$-prime submodule of L. By induction on n, there exist an integer $2 \leq k \leq n$ and a basis $\{m_2, \ldots, m_n\}$ of L such that $K \cap L = \mathfrak{M}m_2 + \cdots + \mathfrak{M}m_k + Rm_{k+1} + \cdots + Rm_n$. It follows that $K = Rm_1 + \mathfrak{M}m_2 + \ldots + \mathfrak{M}m_k + Rm_{k+1} + \cdots + Rm_n$, where $\{m_2, \ldots, m_k, m_1, m_{k+1}, \ldots, m_n\}$ is a basis of M. ∎

3. Dimension over Prüfer domains

It is well known that the Krull dimension of R is the supremum of the Krull dimension of $R_{\mathfrak{P}}$, where the supremum is taken over all maximal ideals (or prime ideals) $\mathfrak{P}$ of R. An analogous result holds for dimensions of modules. The following was proved in [8, Proposition 1] and [9, Section 2].

PROPOSITION 3.1: *Let $\mathfrak{P}$ be a prime ideal of the domain R and M be an R-module. Then $Q \mapsto Q_{\mathfrak{P}}$ is a bijective inclusion preserving map from the set $\{Q : Q$ is a $\mathfrak{Q}$-prime submodule of M with $\mathfrak{Q} \subseteq \mathfrak{P}\}$ to the set of prime $R_{\mathfrak{P}}$-submodules of the $R_{\mathfrak{P}}$-module $M_{\mathfrak{P}}$. The inverse map is $L \mapsto L \bigcap M$.*

PROPOSITION 3.2: *For any module M over the domain R,*

$$D(M) = \sup\{D(M_{\mathfrak{M}}) : \mathfrak{M} \text{ is a maximal ideal of } R\}.$$

Proof: Follows immediately from Proposition 3.1. ∎

Note that Proposition 3.2 remains valid if maximal ideals are replaced by prime ideals.

LEMMA 3.3: *Let R be a valuation domain, n be a positive integer and M be the free R-module $R^{(n)}$. Then $D(M) = D(R) + n - 1$.*

Proof: By Proposition 2.1, $D(M) \geq D(R) + n - 1$. Thus it remains to prove that $D(M) \leq D(R) + n - 1$. If $D(R) = \infty$, then there is nothing to prove. For the rest of this proof, we suppose that $D(R) < \infty$.

If $D(R) = 0$, then R is a field and $D(M) = \dim_R M - 1 = n - 1$, because in this case every proper submodule of M is prime.

Suppose that $D(R) = d > 0$ and the result holds for all valuation domains of smaller dimensions. If $n = 1$, then $M \cong R$ and hence $D(M) = D(R) \leq D(R) + n - 1$. Suppose that $n > 1$ and the result holds for all free R-modules of smaller rank. Let

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_t$$

be a chain of prime submodules of M. Let $P = P_1$ and $\mathfrak{P} = (P : M)$. Suppose that $\mathfrak{P} \neq 0$. Then $D(R/\mathfrak{P}) \leq d - 1$ and $M/\mathfrak{P}M \cong (R/\mathfrak{P})^{(n)}$. Moreover, $\mathfrak{P}M \subseteq P_1$ and

$$P_1/\mathfrak{P}M \subsetneq P_2/\mathfrak{P}M \subsetneq \cdots \subsetneq P_t/\mathfrak{P}M$$

is a chain of prime submodules of the free $R/\mathfrak{P}$-module $M/\mathfrak{P}M$. By induction on d, $t - 1 \leq d - 1 + n - 1$ and hence $t \leq d + n - 1$.

Now suppose that $\mathfrak{P} = 0$. Then M/P is a finitely generated torsion-free module over a valuation domain. By [13, Theorem 4.32], M/P is projective. Then $M = P \oplus P'$ for some submodule P' of M. Note that P' is a finitely generated projective module over the quasi-local ring R, so that (see [13, Theorem 4.44]) P'

is free. Since $P \neq 0$, it follows that $P' \cong R^{(n')}$ for some natural number $n' < n$. Next $P_i = P \oplus (P_i \bigcap P')$ for each $1 \leq i \leq t$ and

$$P_1 \bigcap P' \subsetneq P_2 \bigcap P' \subsetneq \cdots \subsetneq P_t \bigcap P'$$

is a chain of prime submodules of P'. By induction on n,

$$t - 1 \leq D(P') \leq D(R) + n' - 1 < D(R) + n - 1,$$

and hence $t \leq D(R) + n - 1$.

In any case, $t \leq D(R) + n - 1$. It follows that $D(M) \leq D(R) + n - 1$. ∎

The next result has been proved by Azizi and Sharif [2, Theorem 3.1] for Dedekind domains.

THEOREM 3.4: *Let R be a Prüfer domain and M be a finitely generated torsion-free R-module. Then $D(M) = D(R) + \operatorname{rk} M - 1$. In particular, $D(R^{(n)}) = D(R) + n - 1$.*

Proof: Note that, as R is a domain, $\operatorname{rk}_R M = \operatorname{rk}_{R_{\mathfrak{M}}} M_{\mathfrak{M}}$ for any maximal ideal $\mathfrak{M}$ of R. In view of Proposition 3.2, it suffices to show the equality holds when R is replaced by $R_{\mathfrak{M}}$, for any maximal ideal $\mathfrak{M}$. Recall that the localization of any Prüfer domain at a maximal ideal is a valuation domain (see [6, Theorem 64]). By Lemma 3.3, $D(M_{\mathfrak{M}}) = D(R_{\mathfrak{M}}) + \operatorname{rk} M_{\mathfrak{M}} - 1$, as required. ∎

COROLLARY 3.5: *Let R be a Prüfer domain and M be any finitely generated R-module. Then $D(M) \leq D(R) + \mu(M) - 1$.*

Proof: This follows from Theorem 3.4 and the argument used in the proof of Corollary 2.8. ∎

Theorem 3.4 shows that in general there is no relation between $D(M)$ and the (Gabriel–Rentschler) Krull dimension of M, for a given module M. For example, let R be a Dedekind domain (which is not a field) and let n be a positive integer. By Theorem 3.4, $D(R^{(n)}) = n$ but $R^{(n)}$ has Krull dimension 1.

THEOREM 3.6: *The following statements are equivalent for a Noetherian domain R.*

(i) *R is a Dedekind domain.*
(ii) *$D(R^{(n)}) = D(R) + n - 1$ for any positive integer n.*
(iii) *$D(M) \leq D(R) + \mu(M) - 1$ for any finitely generated R-module M.*

(iv) $D(R \oplus R) \leq D(R) + 1$.

In Theorem 3.6, it is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). We need a series of lemmas for the proof of (iv) $\Rightarrow$ (i).

LEMMA 3.7: *For any domain R,*

$$\operatorname{ht}\mathfrak{P} + D(R/\mathfrak{P} \oplus R/\mathfrak{P}) \leq D(R \oplus R),$$

where $\mathfrak{P}$ is a prime ideal of R.

Proof: Note that if $\mathfrak{Q}$ is a prime ideal of R, then $\mathfrak{Q} \oplus \mathfrak{Q}$ is a $\mathfrak{Q}$-prime submodule of $R \oplus R$. Also, any prime submodule of $R/\mathfrak{P} \oplus R/\mathfrak{P}$ is of the form $Q/(\mathfrak{P} \oplus \mathfrak{P})$, where Q is a $\mathfrak{Q}$-prime submodule of $R \oplus R$ with $\mathfrak{P} \subseteq \mathfrak{Q}$. The required result follows easily. ∎

LEMMA 3.8: *The following statements are equivalent for any domain R.*

(i) *R is a one-dimensional Prüfer domain.*

(ii) *$D(R \oplus R) \leq 2$.*

Proof: (i) implies (ii) follows from Theorem 3.4. We now show (ii) implies (i) by a contrapositive argument, which was illustrated in [10, Example 3.3]. Suppose that R is not a Prüfer domain. Then there exist elements a, b of R such that $(Ra : Rb) + (Rb : Ra) \neq R$. Let $\mathfrak{M}$ be a maximal ideal of R such that $(Rb : Ra) + (Ra : Rb) \subseteq \mathfrak{M}$. By Proposition 2.10, $D(R \oplus R) \geq 3$. It follows that (ii) implies (i). ∎

LEMMA 3.9: *Suppose that $(R, \mathfrak{M})$ is a regular local ring of (Krull) dimension 2. Then $D(R \oplus R) \geq 4$.*

Proof: Since R is regular of dimension 2, there exists an R-sequence a, b with $\mathfrak{M} = Ra + Rb$ and the quotient rings R/Rb, R/Ra are one-dimensional regular local rings (see [6, Theorem 161]), hence they are DVRs. It is well known that a regular local ring is a UFD.

CLAIM: *$\mathfrak{M}^2$ contains a prime element.*

JUSTIFICATION: Clearly, $a^3 + b^2$ is in $\mathfrak{M}^2$. We now show that it is prime. For suppose not. Then $a^3 + b^2 = xy$ for some elements x, y in $\mathfrak{M}$. By passing to the ring R/Rb and multiplying x by a unit u (and y by the inverse of u) we can suppose without loss of generality that $x = a^2 + rb$ and $y = a + sb$ for some

r, s in R. Now $a^3 + b^2 = (a^2 + rb)(a + sb) = a^3 + rab + sa^2b + rsb^2$. Hence $b = ra + sa^2 + rsb$. Now, as a, b is an R-sequence $1 - rs$ belongs to Ra and $r + sa$ belongs to Rb so that 1 belongs to $Ra + Rb = \mathfrak{M}$, a contradiction. Thus $a^3 + b^2$ is prime. The claim has been justified.

We now construct a chain of prime submodules of length 4 in $R \oplus R$. By the above claim, $\mathfrak{M}^2$ contains a prime element p. As a, b is an R-sequence, $R(a, b)$, the cyclic submodule of $R \oplus R$ generated by (a, b), is a 0-prime submodule of $R \oplus R$. Define

$$\Lambda_{pR}(a, b) = \{(x, y) \in R \oplus R : xb - ya \in pR\}.$$

It is easily verified that $\Lambda_{pR}(a, b)$ is a pR-prime submodule of $R \oplus R$. As $(p, 0)$ is in $\Lambda_{pR}(a, b)$ but not in $R(a, b)$, $R(a, b)$ is strictly contained in $\Lambda_{pR}(a, b)$.

We now show that $\Lambda_{pR}(a, b)$ is strictly contained in $\mathfrak{M} \oplus \mathfrak{M}$. Suppose that $(x, y) \in \Lambda_{pR}(a, b) \backslash (\mathfrak{M} \oplus \mathfrak{M})$. Without loss of generality, we may assume that x is a unit. Then $b \in Rp + Ra$. It follows that $\mathfrak{M} = Ra + Rb = Ra + Rp$. As p is in $\mathfrak{M}^2$, we get $\mathfrak{M} = Ra + \mathfrak{M}^2$. By Nakayama's Lemma, $\mathfrak{M} = Ra$, which contradicts R is of dimension 2. Therefore, $\Lambda_{pR}(a, b)$ is contained in $\mathfrak{M} \oplus \mathfrak{M}$. As $(a, 0)$ is in $\mathfrak{M} \oplus \mathfrak{M}$ and not in $\Lambda_{pR}(a, b)$, the above inclusion is strict. From the above argument,

$$0 \subsetneq R(a, b) \subsetneq \Lambda_{pR}(a, b) \subsetneq \mathfrak{M} \oplus \mathfrak{M} \subsetneq R \oplus \mathfrak{M}$$

is a chain of prime submodules of length 4 in $R \oplus R$. ∎

In fact, it could be shown that $D(R \oplus R) = 4$ in Lemma 3.9.

THEOREM 3.10: *Suppose that R is a Noetherian domain with $D(R \oplus R) \leq D(R) + 1$. Then R is a Dedekind domain.*

Proof: By Proposition 3.2, we may assume that R is a local Noetherian domain of dimension at least one with maximal ideal $\mathfrak{M}$. If $D(R) = 1$, then by Lemma 3.8, we are done. To finish off the proof, we first show that $D(R) = 2$ is not possible and then use it to deduce that $D(R)$ can not be greater than 2.

Suppose that $D(R) = 2$. By assumption, $D(R \oplus R) \leq 3$. It follows from Lemma 3.9 that R is not regular. Thus R has a height one prime ideal $\mathfrak{P}$ such that $R_{\mathfrak{P}}$, the localization of R at $\mathfrak{P}$, is not a DVR. Hence there exist a, b in $\mathfrak{P}$ with $b \in \mathfrak{P}R_{\mathfrak{P}} \backslash (aR_{\mathfrak{P}})$ and $a \in \mathfrak{P}R_{\mathfrak{P}} \backslash (bR_{\mathfrak{P}})$. Then $(Ra : Rb) + (Rb : Ra) \subseteq \mathfrak{P}$. By Proposition 2.10, $D(R \oplus R) \geq 4 + D(R/\mathfrak{P}) - 1 \geq 4$. This contradicts $D(R \oplus R) \leq 3$. Therefore $D(R \oplus R)$ cannot equal 2.

Suppose that $D(R) \geq 3$. Put $n = D(R)$. Let

$$0 \subsetneq \mathfrak{P}_1 \subsetneq \mathfrak{P}_2 \subsetneq \cdots \subsetneq \mathfrak{P}_{n-2} \subsetneq \mathfrak{P}_{n-1} \subsetneq \mathfrak{P}_n = \mathfrak{M}$$

be a chain of prime ideals of length n in R. Let $\bar{R} = R/\mathfrak{P}_{n-2}$. Note that $\operatorname{ht}\mathfrak{P}_{n-2} + D(\bar{R}) = n$, $\operatorname{ht}\mathfrak{P}_{n-2} = n-2$ and $D(\bar{R}) = 2$.

By Lemma 3.7 and assumption

$$\operatorname{ht}\mathfrak{P}_{n-2} + D(\bar{R} \oplus \bar{R}) \leq D(R \oplus R) \leq D(R) + 1.$$

Hence $\operatorname{ht}\mathfrak{P}_{n-2} + D(\bar{R} \oplus \bar{R}) \leq (\operatorname{ht}\mathfrak{P}_{n-2} + D(\bar{R})) + 1$, i.e., $D(\bar{R} \oplus \bar{R}) \leq D(\bar{R}) + 1$ with $D(\bar{R}) = 2$. We have seen earlier that this is not possible. Thus $D(R) \geq 3$ is not possible. ∎

Proof of Theorem 3.6: (i) $\Rightarrow$ (ii) by Theorem 3.4. (ii) $\Rightarrow$ (iii) follows from the proof of Corollary 2.8. It is clear that (iii) $\Rightarrow$ (iv). By Theorem 3.10, (iv) $\Rightarrow$ (i). ∎

Note that in the above proof, we only need R to be Noetherian in the implication (iv) $\Rightarrow$ (i). It is not known whether Theorem 3.10 holds without R being Noetherian.

4. Dimension of finitely generated modules over Dedekind domains

We want to calculate $D(M)$ in case M is a finitely generated module over a Dedekind domain. In view of Proposition 3.2, in calculating $D(M)$ we can reduce to the case when R is quasi-local, i.e., R has a unique maximal ideal. If I is an ideal of an arbitrary ring R, then an R-module X is called **I-torsion** if, for each $x \in X$, there exists a positive integer n such that $I^n x = 0$.

LEMMA 4.1: *Let R be a quasi-local ring with unique maximal ideal $\mathfrak{M}$ and let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2, where M_2 is $\mathfrak{M}$-torsion. Let $\mathfrak{P}$ be a non-maximal prime ideal of R. Then L is a $\mathfrak{P}$-prime submodule of M if and only if $L = K \oplus M_2$ for some $\mathfrak{P}$-prime submodule K of M_1.*

Proof: Let L be a $\mathfrak{P}$-prime submodule of M for some prime ideal $\mathfrak{P} \neq \mathfrak{M}$. For each $x \in M_2$, there exists a positive integer n such that $\mathfrak{M}^n x = 0 \subseteq L$, so that $x \in L$. Hence $M_2 \subseteq L$ and $L = K \oplus M_2$, where $K = L \cap M_1$. Since $M_1/K \cong M/L$, it follows that K is a $\mathfrak{P}$-prime submodule of M_1.

Conversely, if $L = K \oplus M_2$ for some $\mathfrak{P}$-prime submodule K of M_1, then $M/L \cong M_1/K$ gives that L is a $\mathfrak{P}$-prime submodule of M. ∎

LEMMA 4.2: *Let R be a quasi-local ring with unique maximal ideal $\mathfrak{M}$ and let an R-module $M = M_1 \oplus M_2$ be a direct sum of finitely generated submodules M_1, M_2, where M_2 is $\mathfrak{M}$-torsion. Then $D(M) = \max\{D(M_1), \mu(M) - 1\}$.*

Proof: Because M_1 is a homomorphic image of M we see that $D(M_1) \leq D(M)$. Moreover, $D(M) \geq D(M/\mathfrak{M}M) = \mu(M/\mathfrak{M}M) - 1$. But $\mu(M/\mathfrak{M}M) = \mu(M)$ by Nakayama's Lemma. Hence $D(M) \geq \mu(M) - 1$. Let

$$L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_t$$

be any chain of prime submodules of M, for some non-negative integer t. If $(L_0 : M) = \mathfrak{M}$, then $t \leq \mu(M) - 1$. Suppose that $(L_0 : M) = \mathfrak{P}$ for some prime ideal $\mathfrak{P} \neq \mathfrak{M}$. By Lemma 4.1, $L_0 = K_0 \oplus M_2$ for some $\mathfrak{P}$-prime submodule K_0 of M_1. Moreover, $L_i = K_i \oplus M_2$ for some prime submodule K_i of M_1 for each $1 \leq i \leq t$. Clearly,

$$K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_t$$

is a chain of prime submodule of M_1. Hence $t \leq D(M_1)$. It follows that $D(M) \leq \max\{D(M_1), \mu(M) - 1\}$. ∎

THEOREM 4.3: *Let R be a domain, let n be a positive integer, let $\mathfrak{P}_i$ $(1 \leq i \leq n)$ be distinct maximal ideals of R and let an R-module $M = M' \oplus M_1 \oplus \cdots \oplus M_n$ be a direct sum of a finitely generated torsion-free submodule M' and finitely generated $\mathfrak{P}_i$-torsion submodules $M_i (1 \leq i \leq n)$. Then $D(M) = \max\{D(M'), \mu(M/\mathfrak{P}_1 M) - 1, \ldots, \mu(M/\mathfrak{P}_n M) - 1\}$.*

Proof: Let $\mathfrak{P}$ be any maximal ideal of R. Suppose that $\mathfrak{P} \neq \mathfrak{P}_i$ $(1 \leq i \leq n)$. Then $M_{\mathfrak{P}} = M'_{\mathfrak{P}}$ and hence $D(M_{\mathfrak{P}}) = D(M'_{\mathfrak{P}}) \leq D(M')$ by Proposition 3.2. Now suppose that $\mathfrak{P} = \mathfrak{P}_i$ for some $1 \leq i \leq n$. Then $M_{\mathfrak{P}} \cong M'_{\mathfrak{P}} \oplus M_i$ and hence $D(M_{\mathfrak{P}}) = \max\{D(M'_{\mathfrak{P}}), \mu(M_{\mathfrak{P}}/\mathfrak{P}M_{\mathfrak{P}}) - 1\}$ by Lemma 4.2. But $D(M'_{\mathfrak{P}}) \leq D(M')$ and $M_{\mathfrak{P}}/\mathfrak{P}M_{\mathfrak{P}} \cong M/\mathfrak{P}M$. By Proposition 3.2, we have proved that $D(M) \leq \max\{D(M'), \mu(M/\mathfrak{P}_1 M) - 1, \ldots, \mu(M/\mathfrak{P}_n M) - 1\}$.

Conversely, $D(M') \leq D(M)$ because M' is a homomorphic image of M. Moreover, for each $1 \leq i \leq n$, $\mu(M/\mathfrak{P}_i M) - 1 = D(M/\mathfrak{P}_i M) \leq D(M)$. The result follows. ∎

COROLLARY 4.4: *Let R be a domain, let n be a positive integer, let $\mathfrak{P}_i$ $(1 \leq i \leq n)$ be distinct maximal ideals of R and let an R-module $M = M' \oplus M_1 \oplus \cdots \oplus M_n$*

be a direct sum of a free submodule M' of finite rank k and finitely generated $\mathfrak{P}_i$-torsion submodules $M_i (1 \le i \le n)$. Then

$$D(M) = \max\{D(M'), \mu(M_1) + k - 1, \dots, \mu(M_n) + k - 1\}.$$

Proof: For each $1 \le i \le n$, $M/\mathfrak{P}_i M \cong (R/\mathfrak{P}_i)^{(k)} \oplus (M_i/\mathfrak{P}_i M_i)$ and hence $\mu(M/\mathfrak{P}_i M) = k + \mu(M_i/\mathfrak{P}_i M_i)$. Since M_i is $\mathfrak{P}_i$-torsion, it follows that $\mu(M_i/\mathfrak{P}_i M_i) = \mu(M_i)$. Apply Theorem 4.3. ∎

We now apply Theorem 4.3 and Corollary 4.4 to find $D(M)$ in case M is a finitely generated module over a Dedekind domain.

THEOREM 4.5: *Let R be a Dedekind domain and let M be a finitely generated R-module. Then $M = M' \oplus M_1 \oplus \cdots \oplus M_n$ is a direct sum of a torsion-free submodule M' and $\mathfrak{P}_i$-torsion submodules $M_i (1 \le i \le n)$ for some positive integer n and distinct maximal ideals $\mathfrak{P}_i (1 \le i \le n)$. Moreover,*

$$D(M) = \max\{\operatorname{rk} M' + \mu(M_i) - 1 : 1 \le i \le n\}.$$

Proof: The first part is well known. If $M' = 0$, then $\operatorname{rk} M' = 0$ and the result follows by Corollary 4.4. Suppose that $M' \neq 0$. By [4, Theorem 6.11], there exist a positive integer k and an ideal I of R such that $M' \cong R^{(k-1)} \oplus I$. For any maximal ideal $\mathfrak{P}$ of R, $I/\mathfrak{P}I \cong R/\mathfrak{P}$ (see [4, Corollary 6.5]) and $M'/\mathfrak{P}M' \cong (R/\mathfrak{P})^{(k)}$. By the proof of Corollary 4.4, $\mu(M/\mathfrak{P}_i M) = k + \mu(M_i) = \operatorname{rk} M' + \mu(M_i)$ for all $1 \le i \le n$. Also, Theorem 3.4 gives that $D(M') = \operatorname{rk} M'$. By Theorem 4.3, $D(M) = \max\{\operatorname{rk} M' + \mu(M_i) - 1 : 1 \le i \le n\}$. ∎

Next we prove that if R is a Dedekind domain, then the lower bound in Proposition 2.2 is attained.

COROLLARY 4.6: *Let R be a Dedekind domain and let M be a finitely generated R-module. Then $D(M) = \sup\{D(R/\mathfrak{P}) + \operatorname{rk}_{\mathfrak{P}} M - 1 : \mathfrak{P}$ is a prime ideal of R and $\operatorname{ann}(M) \subseteq \mathfrak{P}\}$.*

Proof: As in Theorem 4.5, $M = M' \oplus M_1 \oplus \cdots \oplus M_n$ is a direct sum of a torsion-free submodule M' and $\mathfrak{P}_i$-torsion submodules M_i $(1 \le i \le n)$, for some positive integer n and distinct maximal ideals $\mathfrak{P}_i (1 \le i \le n)$. Suppose that $M' = 0$. Let $\mathfrak{P}$ be any prime ideal of R such that $\operatorname{ann}(M) \subseteq \mathfrak{P}$. Then $\mathfrak{P} = \mathfrak{P}_i$ for some $1 \le i \le n$. In this case, $D(R/\mathfrak{P}) + \operatorname{rk}_{\mathfrak{P}} M - 1 = 0 + \mu(M/\mathfrak{P}M) - 1$. By Theorem 4.3,

$$D(M) = \sup\{D(R/\mathfrak{P}) + \operatorname{rk}_{\mathfrak{P}} M - 1 : \mathfrak{P} \text{ is a prime ideal of } R \text{ and } \operatorname{ann}(M) \subseteq \mathfrak{P}\}.$$

Now suppose that $M' \neq 0$. In this case, $\mathrm{ann}(M) = 0$. Let $\mathfrak{P}$ be a prime ideal of R. If $\mathfrak{P} = 0$, then $D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}}(M) - 1 = \mathrm{rk}\, M'$, because $D(R) = 1$. Suppose that $\mathfrak{P} = \mathfrak{P}_i$ for some $1 \leq i \leq n$. Then $D(R/\mathfrak{P}) = 0$ and $\mathrm{rk}_{\mathfrak{P}}\, M = \mu(M/\mathfrak{P}M)$. Finally, suppose that $\mathfrak{P}$ is a maximal ideal of R such that $\mathfrak{P} \neq \mathfrak{P}_i$ $(1 \leq i \leq n)$. Then $D(R/\mathfrak{P}) = 0$ and $M/\mathfrak{P}M \cong M'/\mathfrak{P}M'$, so that $\mathrm{rk}_{\mathfrak{P}}\, M = \mathrm{rk}\, M'$. We have proved that, for any prime ideal $\mathfrak{P}$ of R,

$$D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}}\, M - 1 = \begin{cases} \mathrm{rk}\, M' & \text{if } \mathfrak{P} = 0, \\ \mu(M/\mathfrak{P}M) - 1 & \text{if } \mathfrak{P} = \mathfrak{P}_i \text{ for some } 1 \leq i \leq n, \\ \mathrm{rk}\, M' - 1 & \text{otherwise.} \end{cases}$$

But $\mathrm{rk}\, M' = D(M')$ by Theorem 3.4. By Theorem 4.3, $D(M) = \sup\{D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}}\, M - 1 : \mathfrak{P}$ is a prime ideal of R and $\mathrm{ann}(M) \subseteq \mathfrak{P}\}$. ∎

THEOREM 4.7: *Suppose that R is a Noetherian domain. Then R being Dedekind is equivalent to*

$$D(M) = \sup\{D(R/\mathfrak{P}) + \mathrm{rk}_{\mathfrak{P}}\, M - 1 : \mathfrak{P} \textit{ is a prime ideal of } R \textit{ and } \mathrm{ann}(M) \subseteq \mathfrak{P}\},$$

for any finitely generated R-module M.

Proof: The necessary condition is just Corollary 4.6. For the converse, put $M = R^{(2)}$. Then $D(M) = D(R) + 1$. Now, R is Dedekind follows from Theorem 3.6. ∎

5. Dimension of free modules

Let R be a Noetherian one-dimensional domain and $\mathfrak{P}$ be any maximal ideal of R. The local domain $R_{\mathfrak{P}}$ is one-dimensional and Cohen [3, Theorem] proved that there exists a positive integer k such that every ideal of $R_{\mathfrak{P}}$ can be generated by k elements.

Recall that from section 1, $\nu(R_{\mathfrak{P}})$ is defined to be the least positive integer k such that every ideal of $R_{\mathfrak{P}}$ can be generated by k elements. Also, we define

$$\nu(R) = \sup\{\nu(R_{\mathfrak{P}}) : \mathfrak{P} \text{ is a maximal ideal of } R\}.$$

Note that $\nu(R)$ is a positive integer or $\nu(R) = \infty$. In [3, pp. 39–40], an example is given of a Noetherian one-dimensional domain R such that $\nu(R) = \infty$. Note that a Noetherian domain R is Dedekind if and only if R is one-dimensional and $\nu(R) = 1$.

For any real number r, we let $[r]$ denote the greatest integer s such that $s \leq r$. Given an integer t, we write $\nu(R) \mid t$ if $\nu(R)$ is a positive integer and $\nu(R)$ divides

t; otherwise, we write $\nu(R) \nmid t$. Moreover, if $\nu(R) = \infty$, we define $[n/\nu(R)] = 0$ for any positive integer n. We shall prove Theorem 5.1 in a series of lemmas.

THEOREM 5.1: *Let R be a Noetherian one-dimensional domain and let n be a positive integer. Then*

$$D(R^{(n)}) = \begin{cases} 2n - n/\nu(R) & \text{if } \nu(R) \mid n, \\ 2n - [n/\nu(R)] - 1 & \text{if } \nu(R) \nmid n. \end{cases}$$

We deal first with the case $\nu(R) = \infty$.

LEMMA 5.2: *Let R be a Noetherian one-dimensional domain. Then, $\nu(R) \geq n$ if and only if $D(R^{(n)}) = 2n - 1$.*

Proof: Let $M = R^{(n)}$. By Theorem 2.7, $D(M) \leq 2n-1$. Suppose that $\nu(R) \geq n$. Then there exists a maximal ideal $\mathfrak{P}$ of R such that $\nu(R_{\mathfrak{P}}) = k \geq n$. Let I be an ideal of $R_{\mathfrak{P}}$ such that I can be generated by k, but no fewer, elements. Then $I = R_{\mathfrak{P}}a_1 + R_{\mathfrak{P}}a_2 + \cdots + R_{\mathfrak{P}}a_k$ for some elements a_i $(1 \leq i \leq k)$ of R. Let

$$K = \{(r_1, r_2, \ldots, r_n) \in M : a_1r_1 + a_2r_2 + \cdots + a_nr_n = 0\}.$$

By Lemma 2.9, K is a 0-prime submodule of M and $\operatorname{ht} K = n - 1$. Moreover, if $r_i \in R$ $(1 \leq i \leq n)$ and $a_1r_1 + a_2r_2 + \cdots + a_nr_n = 0$, then r_i is not a unit in $R_{\mathfrak{P}}$ and hence $r_i \in \mathfrak{P}$ for $1 \leq i \leq n$; that is, $K \subseteq \mathfrak{P}M$. It follows that

$$\begin{aligned} D(M) \geq & D(M/\mathfrak{P}M) + \operatorname{ht}(\mathfrak{P}M), \\ \geq & D(M/\mathfrak{P}M) + 1 + \operatorname{ht} K, \\ \geq & (n-1) + 1 + (n-1) = 2n - 1. \end{aligned}$$

Thus $D(M) = 2n - 1$.

Conversely, suppose that $D(M) = 2n - 1$. Let

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_{2n-1}$$

be a chain of prime submodules of M. From Lemma 2.6 and $D(M) = 2n - 1$,

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_{n-1}$$

is a homogeneous chain of 0-prime submodules of M; and

$$P_n \subsetneq P_{n+1} \subsetneq P_{n+2} \subsetneq \cdots \subsetneq P_{2n-1}$$

is a homogeneous chain of $\mathfrak{M}$-prime submodules of M, where $\mathfrak{M}$ is a maximal ideal. As the length of any homogeneous chain of $\mathfrak{M}$-prime submodules of M is at most $n - 1$, we have $P_n = \mathfrak{M}M$.

By Lemma 2.9, there exist elements $a_i \in R (1 \leq i \leq n)$ and a basis $\{u_1, u_2, \ldots, u_n\}$, which we may assume to be the standard basis, of M such that

$$P_{n-1} = \{(r_1, r_2, \ldots, r_n) \in M : a_1 r_1 + a_2 r_2 + \cdots + a_n r_n = 0\}.$$

Fix an integer i between 1 and n. Let $I_i = Ra_1 + \cdots + Ra_{i-1} + Ra_{i+1} + \cdots + Ra_n$ and $r \in (I_i : Ra_i)$. Then

$$ra_i = s_1 a_1 + \cdots + s_{i-1} a_{i-1} + s_{i+1} a_{i+1} + \cdots + s_n a_n$$

for some $s_j \in R$ $(1 \leq j \leq n, j \neq i)$. It follows that

$$(s_1, \ldots, s_{i-1}, -r, s_{i+1}, \ldots, s_n) \in P_{n-1} \subseteq \mathfrak{M}M,$$

so that $r \in \mathfrak{M}$. Hence $\sum_{i=1}^{n} (I_i : Ra_i) \subseteq \mathfrak{M}$. Let $I = Ra_1 + Ra_2 + \cdots + Ra_n$. It is easy to check that $I_{\mathfrak{M}}$ can be generated by n, and no fewer, elements over $R_{\mathfrak{M}}$. It follows that $\nu(R) \geq \nu(R_{\mathfrak{M}}) \geq n$. ∎

Note that in proving Lemma 5.2, we do not need R to be Noetherian. In view of Lemma 5.2, Theorem 5.1 is true for $\nu(R) = \infty$. We now suppose that $\nu(R)$ is a positive integer. The next step is to reduce to the local case. It will be convenient to call temporarily a Noetherian one-dimensional domain R **good** if R satisfies the conclusion of Theorem 5.1.

Lemma 5.3: *Suppose that R is a Noetherian one-dimensional domain with $\nu(R) < \infty$ and the local domain $R_{\mathfrak{P}}$ is good for every maximal ideal $\mathfrak{P}$ of R. Then R is good.*

Proof: Suppose that $\nu(R) = k$ (so that k is a positive integer). There exists a maximal ideal $\mathfrak{P}$ of R such that $\nu(R_{\mathfrak{P}}) = k$ and $\nu(R_{\mathfrak{Q}}) \leq k$ for every maximal ideal $\mathfrak{Q}$. Suppose that $k \mid n$. Let $M = R^{(n)}$. By hypothesis, $D(M_{\mathfrak{P}}) = 2n - n/k$. Suppose that there exists a maximal ideal $\mathfrak{Q}$ of R such that $\nu(R_{\mathfrak{Q}}) < k$. Then $n/k < n/\nu(R_{\mathfrak{Q}})$ and hence $n/k \leq [n/\nu(R_{\mathfrak{Q}})]$. By hypothesis,

$$D(M_{\mathfrak{Q}}) = \begin{cases} 2n - n/\nu(R_{\mathfrak{Q}}) & \text{if } \nu(R_{\mathfrak{Q}}) \mid n, \\ 2n - [n/\nu(R_{\mathfrak{Q}})] - 1 & \text{if } \nu(R_{\mathfrak{Q}}) \nmid n. \end{cases}$$

Hence $D(M_{\mathfrak{Q}}) \leq 2n - n/k$. By Proposition 3.2, $D(M) = 2n - n/k$, as required.

Now suppose that $k \nmid n$. By hypothesis, $D(M_{\mathfrak{P}}) = 2n - [n/k] - 1$. Again let $\mathfrak{Q}$ be a maximal ideal of R such that $\nu(R_{\mathfrak{Q}}) < k$. Then $[n/k] < n/k < n/\nu(R_{\mathfrak{Q}})$. But, by hypothesis,

$$D(M_{\mathfrak{Q}}) = \begin{cases} 2n - n/\nu(R_{\mathfrak{Q}}) & \text{if } \nu(R_{\mathfrak{Q}}) \mid n, \\ 2n - [n/\nu(R_{\mathfrak{Q}})] - 1 & \text{if } \nu(R_{\mathfrak{Q}}) \nmid n. \end{cases}$$

Hence $D(M_{\mathfrak{Q}}) \leq 2n - [n/k] - 1$. By Proposition 3.2, $D(M) = 2n - [n/k] - 1$. ∎

We shall need the following result to prove Theorem 5.1 holds for local domains.

LEMMA 5.4: *Let R be a Noetherian one-dimensional local domain with unique maximal ideal $\mathfrak{M}$ and $\nu(R) = k < \infty$. Let n be a positive integer. Write $n = kq + r$ for some non-negative integers q, r with $0 \leq r < k$. Let K be a 0-prime submodule of $R^{(n)}$ with $K \subseteq \mathfrak{M}R^{(n)}$. Then $\operatorname{ht} K \leq n - q$, if $k \mid n$, and $\operatorname{ht} K \leq n - q - 1$, otherwise. Furthermore, the maximums are attainable.*

Proof: Let $M = R^{(n)}$. Suppose that $k = 1$. Then R is a DVR and $n = q$. We have $\operatorname{ht} K + 1 + D(M/\mathfrak{M}M) \leq D(M)$. As $M/\mathfrak{M}M$ is an n-dimensional vector space over $R/\mathfrak{M}$, $D(M/\mathfrak{M}M) = n - 1$. Since R is a DVR, by Theorem 3.4, $D(M) = n$. It follows that $\operatorname{ht} K = 0$. We have shown that $\operatorname{ht} K \leq n - q$ when $k = 1$.

Suppose that $k \geq 2$ and $n < k$. Then $q = 0$. We have $\operatorname{ht} K + 1 + D(M/\mathfrak{M}M) \leq D(M)$. By Theorem 2.7, $D(M) \leq 2n - 1$. Hence $\operatorname{ht} K + 1 + n - 1 \leq 2n - 1$, which gives $\operatorname{ht} K \leq n - 1$.

From now on we may assume that $k \geq 2$ and $n \geq k$. By Lemma 2.9, we can suppose without loss of generality that there exists an $m \times n$ matrix A over R, for some integer $1 \leq m \leq n$, such that

$$K = \{(r_1, r_2, \ldots, r_n) \in R^{(n)} : A \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = 0\}$$

and $\operatorname{ht} K = n - m$.

We now show that if $k \mid n$, then $\operatorname{ht} K \leq n - q$. The other case $k \nmid n$ can be proved in a similar fashion. Suppose that $k \mid n$ and $\operatorname{ht} K > n - q$. Then $m < q$. Let I be the ideal generated by all the entries of row 1 of A. As $\nu(R) = k$, I is generated by k of the entries in row 1 of A. Hence there exists an invertible $n \times n$ matrix E_1 over R such that the last $n - k$ entries of row 1 of AE_1 are all zero. In other words, if $AE_1 = (b_{ij})$, then $b_{1j} = 0$ for $k + 1 \leq j \leq n$. Similarly, there exists an invertible $n \times n$ matrix E_2 over R such that

(i) the last $n - k$ entries of row 1 of AE_1E_2 are all zero,

(ii) the last $n - 2k$ entries of row 2 of AE_1E_2 are all zero.

By repeating the above process, there exists an invertible $n \times n$ matrix E over R such that, for $d = 1, 2, \ldots, m$, the last $n - dk$ entries of row d of AE are all zero.

Write $AE = [B \mid C]$, where B is an $m \times (mk)$ matrix and C is an $m \times (n-mk)$ zero matrix. In particular, all the entries in the nth (last) column of AE are zero. Write

$$E \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix},$$

which is the nth column of E. Then $(c_1, c_2, \ldots, c_n)$ is in K. As $K \subseteq \mathfrak{M}R^{(n)}$, $c_i \in \mathfrak{M}$ for all $1 \le i \le n$. This implies that $\det E \in \mathfrak{M}$, which contradicts E is invertible. Therefore $\operatorname{ht} K \le n - q$.

We prove the last assertion by constructing a 0-prime submodule of $R^{(n)}$, which is contained in $\mathfrak{M}R^{(n)}$, of height $n-q$, if $k \mid n$, and of height $n-q-1$, otherwise. As $\nu(R) = k$, there exist $a_1, a_2, \ldots, a_k \in R$, such that $J = Ra_1 + Ra_2 + \cdots + Ra_k$ and J cannot be generated by less than k elements. Then $\sum_{i=1}^{k} (J_i : Ra_i) \subseteq \mathfrak{M}$, where $J_i = \sum_{\substack{j=1 \\ j \ne i}}^{k} Ra_j$ for $1 \le i \le k$. Let

$$\begin{aligned} K = \{&(x_1, x_2, \ldots, x_n) \in R^{(n)} : \\ &a_1 x_{ik+1} + a_2 x_{ik+2} + \cdots + a_k x_{(i+1)k} = 0 (0 \le i \le q-1) \text{ and} \\ &a_1 x_{qk+1} + a_2 x_{qk+2} + \cdots + a_r x_r = 0\}. \end{aligned}$$

Note that $K \subseteq \mathfrak{M}R^{(n)}$. By Lemma 2.9, K is a 0-prime submodule of $R^{(n)}$ and

$$\operatorname{ht} K = \begin{cases} n - q & \text{if } k \mid n, \\ n - q - 1 & \text{if } k \nmid n. \end{cases} \qquad \blacksquare$$

LEMMA 5.5: *With the notation of Lemma 5.4,*

$$D(R^{(n)}) = \begin{cases} 2n - n/\nu(R) & \text{if } \nu(R) \mid n, \\ 2n - [n/\nu(R)] - 1 & \text{if } \nu(R) \nmid n. \end{cases}$$

Proof: Let $M = R^{(n)}$ and $\nu(R) = k < \infty$. By Theorem 2.7, $D(M) \le 2n - 1 < \infty$. Let $D(M) = t$ and let

$$K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_t$$

be a chain of prime submodules $K_i (1 \le i \le t)$ of M. As R is a one-dimensional local domain with unique maximal ideal $\mathfrak{M}$, for each $1 \le i \le t$, K_i is $\mathfrak{M}$-prime or 0-prime. Moreover, K_t is $\mathfrak{M}$-prime, $K_0 = 0$ and K_0 is 0-prime. There exists $0 < s < t$ such that K_i is 0-prime if $0 \le i \le s$ and $\mathfrak{M}$-prime if $s+1 \le i \le t$. Consider the $\mathfrak{M}$-prime submodule K_{s+1}. By Proposition 2.11, there exist a basis

$\{u_1, u_2, \ldots, u_n\}$ and a positive integer $1 \le h \le n$ such that $K_{s+1} = \mathfrak{M}u_1 + \cdots + \mathfrak{M}u_h + Ru_{h+1} + \cdots + Ru_n$. Note that $M/K_{s+1} \cong (R/\mathfrak{M})^{(h)}$, so that $D(M/K_{s+1}) = h - 1$.

Now consider the 0-prime submodule K_s. By Lemma 2.9, there exists an $m \times n$ matrix A over R, for some integer $1 \le m \le n$, such that

$$K_s = \{r_1u_1 + r_2u_2 + \cdots + r_nu_n \in M : A \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = 0\},$$

and $\operatorname{ht} K_s = n - m$.

For $i = 1, 2, \ldots, n-h$, let $Q_i = \mathfrak{M}u_1 + \cdots + \mathfrak{M}u_{h+i} + Ru_{h+i+1} + \cdots + Ru_n$. Clearly, each $Q_i (1 \le i \le n-h)$ is an $\mathfrak{M}$-prime submodule of M with $Q_i \subseteq K_{s+1}$. Note that $Q_{n-h} = \mathfrak{M}M$ and

$$Q_{n-h} \subsetneq Q_{n-(h+1)} \subsetneq \cdots \subsetneq Q_1 \subsetneq K_{s+1}.$$

Hence, $\operatorname{ht} K_{s+1} \ge (n-h) + \operatorname{ht} Q_{n-h}$. Write $n = kq' + r'$, where q', r' are non-negative integers with $0 \le r' < k$. By Lemma 5.4,

$$\operatorname{ht} Q_{n-h} = \begin{cases} n - q' + 1 & \text{if } k \mid n, \\ n - q' & \text{if } k \nmid n. \end{cases}$$

As $s = \operatorname{ht} K_s = n - m$ and $\operatorname{ht} K_{s+1} = s + 1$,

$$m \le \begin{cases} h + (q' - n) & \text{if } k \mid n, \\ h + (q' + 1 - n) & \text{if } k \nmid n. \end{cases}$$

In any case, $m \le h$.

Write $A = [B \mid C]$, where B is an $m \times h$ matrix and C is an $m \times (n-h)$ matrix. Let

$$K = \{r_1u_1 + r_2u_2 + \cdots + r_hu_h \in M : B \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_h \end{pmatrix} = 0\}.$$

By Lemma 2.9, K is a 0-prime submodule of height $h - m$ in $Ru_1 + \cdots + Ru_h$, and K is contained in $\mathfrak{M}u_1 + \cdots + \mathfrak{M}u_h$. Write $h = kq + r$, where q, r are non-negative integers with $0 \le r < k$. By Lemma 5.4, $m \ge q$, if $k \mid h$ and $m \ge q + 1$, otherwise. Hence $\operatorname{ht} K_s \le n - q$, if $k \mid h$ and $\operatorname{ht} K \le n - q - 1$, otherwise. By an argument used in the last part of the proof of Lemma 5.4, we can construct a 0-prime submodule P in M such that

(i) P is contained in $\mathfrak{M}u_1 + \cdots + \mathfrak{M}u_h + Ru_{h+1} + Ru_{h+2} + \cdots + Ru_n$, and
(ii) $\operatorname{ht} P = n - q$, if $k \mid h$, and $\operatorname{ht} P = n - q - 1$, otherwise.

Therefore

$$\operatorname{ht} K_s = \begin{cases} n-q & \text{if } k \mid h, \\ n-q-1 & \text{if } k \nmid h. \end{cases}$$

It follows that

$$t = (h-1) + 1 + \operatorname{ht} K_s = \begin{cases} h+n-h/k & \text{if } k \mid h, \\ h+n-[h/k]-1 & \text{if } k \nmid h. \end{cases}$$

Let $f\colon \{1, 2, \ldots, n\} \longrightarrow \mathbb{N}$ be the mapping defined by

$$f(h) = \begin{cases} h+n-h/k & \text{if } k \mid h, \\ h+n-[h/k]-1 & \text{if } k \nmid h. \end{cases}$$

It is easy to check that f is increasing and hence

$$t = f(n) = \begin{cases} 2n-n/k & \text{if } k \mid n, \\ 2n-[n/k]-1 & \text{if } k \nmid n. \end{cases} \qquad \blacksquare$$

Proof of Theorem 5.1: By Lemmas 5.2, 5.3 and 5.5. ∎

The proof of Theorem 5.1 shows that if R is a Noetherian one-dimensional domain and n is a positive integer, then $D(R^{(n)}) = D((R/\mathfrak{M})^{(n)}) + \operatorname{ht}(\mathfrak{M}R^{(n)})$ for some maximal ideal $\mathfrak{M}$ of R.

We end this section with a characterization of $\nu(R)$.

THEOREM 5.6: *Let R be a Noetherian one-dimensional domain and n be a positive integer. The following statements are equivalent.*

(i) $\nu(R) = n$.
(ii) $D(R^{(k)}) = 2k - 1$ *for* $k = 1, 2, \ldots, n$; *and* $D(R^{(k)}) < 2k - 1$ *for all* $k > n$.

Proof: (i)⇒(ii) By Theorem 5.1.
(ii)⇒(i) Let $m = \nu(R)$. Suppose that $m > n$. Then, by (ii), $D(R^{(m)}) < 2m - 1$, which contradicts Theorem 5.1. Thus $m \leq n$. Suppose that $m < n$. By Theorem 5.1, $D(R^{(n)}) < 2n - 1$, which contradicts (ii). Thus $m = n$. ∎

COROLLARY 5.7: *The following statements are equivalent for a Noetherian one-dimensional domain R.*

(i) $\nu(R) = \infty$.
(ii) $D(R^{(k)}) = 2k - 1$ *for every positive integer* k.

Proof: By Theorems 5.1 and 5.6. ∎

6. An example

In section 2, we promised to give an example of a domain R and a finitely generated R-module M such that $D(M) = \mu(M)D(R) + \mu(M) - 1$. We do this next. Recall that a non-zero integer d is called **square-free** if $d \notin \mathbb{Z}p^2$ for every prime p in $\mathbb{Z}$. Clearly, if d is a square-free integer then $d \not\equiv 0(\text{mod}4)$. Let d be a square-free integer and let R denote the subring $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$ of the field $\mathbb{C}$ of complex numbers. Then R is a Noetherian one-dimensional domain. If $d \equiv 2$ or $3(\text{mod}4)$, then R is a Dedekind domain and $D(R^{(n)}) = n$ for all $n \geq 1$.

THEOREM 6.1: *Let d be a square-free integer such that $d \equiv 1(\text{mod}4)$ and let R be the ring $\mathbb{Z}[\sqrt{d}]$. Then $D(R^{(2)}) = 3$ and $D(R^{(n)}) < 2n - 1$ for all $n \geq 3$.*

Proof: Define $\varphi\colon R \longrightarrow \mathbb{Z}/\mathbb{Z}2$ by $\varphi(a + b\sqrt{d}) = (a + b) + \mathbb{Z}2$ for all $a, b \in \mathbb{Z}$. It is easy to check that φ is a ring epimorphism. If $\mathfrak{M}$ is the maximal ideal $\ker\varphi$, then $\mathfrak{M} = \{a + b\sqrt{d} : a, b \in \mathbb{Z}, a + b \in \mathbb{Z}2\} = R2 + R(1 + \sqrt{d})$. It is also easy to check that

$$(R2 : R(1 + \sqrt{d})) + (R(1 + \sqrt{d}) : R2) \subseteq \mathfrak{M}.$$

Then $\nu(R) \geq \nu(R_{\mathfrak{M}}) \geq 2$. By Lemma 5.2 , $D(R^{(2)}) = 3$.

Suppose that $D(R^{(3)}) = 5$. By Lemma 5.2, there exist elements a_1, a_2, a_3 in R such that

$$(*) \qquad (Ra_2 + Ra_3 : Ra_1) + (Ra_1 + Ra_3 : Ra_2) + (Ra_1 + Ra_2 : Ra_3) \neq R.$$

But $R = \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}$, so that R is a free $\mathbb{Z}$-module of rank 2 and hence there exist $m_i \in \mathbb{Z}(1 \leq i \leq 3)$ such that

$$m_1a_1 + m_2a_2 + m_3a_3 = 0.$$

Without loss of generality, the integers m_1, m_2, m_3 are coprime and hence $\mathbb{Z} = \mathbb{Z}m_1 + \mathbb{Z}m_2 + \mathbb{Z}m_3$. Thus

$$1 \in \mathbb{Z}m_1 + \mathbb{Z}m_2 + \mathbb{Z}m_3 \subseteq \sum_{i=1}^{3}((\sum_{\substack{j=1 \\ j\neq i}}^{3} Ra_j) : Ra_i),$$

and we have $\sum_{i=1}^{3}((\sum_{\substack{j=1 \\ j\neq i}}^{3} Ra_j) : Ra_i) = R$, contradicting $(*)$. Thus $D(R^{(3)}) < 5$. Together with Proposition 5.6, we have $D(R^{(n)}) < 2n - 1$ for all $n \geq 3$. ∎

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